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INERTIAL STEADY FLOW OF A BED OF GRANULAR MATERIAL DOWN A SURFACE WITH MICRORELIEF WITH ALLOWANCE FOR FINITE TIME OF INTERGRANULAR CONTACTING

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An inertial flow of a granular material can be described by the laws of conservation of mass, momentum, and energy of random motion of solid particles by invoking some closing relations. In this work, these closing relations are inferred from the dimensional theory. The system of equations obtained is used to determine characteristics of a steady flow of a bed of a granular material down an inclined surface with a microrelief for various Richardson numbers and finite contact times of the particles during their collisions.

There are many scientific and technological problems concerning the flow of dry granular materials consisting of solid particles. Among these problems are those related to mining raw mineral materials, powder and ceramic technologies, production of new materials, and grain storage and transportation. Granular media are also involved in many geophysical processes, including avalanches.

The fundamentals of the present-day concepts of the mechanics of such media can be found in many reviews and articles (see, e.g., [1] and the bibliography there). Depending on the density of the medium and the shear velocity of the flow, two limiting regimes can be distinguished: 1) quasi-steady flow that corresponds to high concentrations of granules and low shear velocities (the particles are in permanent tight contact, and the behavior of the material can be well described by the Coulomb–Mohr law of dry friction $\tau = \sigma \tan \varphi$ that relates the shear stress τ and the normal stress σ, where φ is the internal-friction angle equal, for example, to 37° for fine sand); 2) inertial flow that corresponds to lower concentrations of granules and higher flow shear velocities [there are always gaps (on the average) between the granules, and the granules interact colliding with one another].

According to [1–3], an inertial flow of a granular material can be described using the laws of conservation of mass, momentum, and energy by invoking some closing relations between the pressure, viscosity, thermal conductivity, and dissipation rate of the energy of random motion of the granules, on the one hand, and the density and energy of the granules, on the other. For the inertial flow, there is a number of closing schemes, for example, schemes based on the kinetic theory of dense gases [4–7] and schemes devised from the dimensional theory [3–8]. It should be noted here that, in the intermediate case of high concentrations and moderate shear velocities, the model of the power-law non-Newtonian fluid with the power exponent $n = 2$ seems to be quite satisfactory since both experiments and estimates show that the stresses and shear velocities are related by a square dependence. In this model, the effect due to intergranular collisions, which can be described by a scalar function (the energy of random motion of the granules), is ignored. In the present work, we use an approach previously proposed in [8, 9] to analyze a steady flow of a bed of a granular material down an inclined surface with a microrelief.

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We consider a steady flow of a bed of a granular material down an inclined surface installed at an angle ψ to the horizontal plane and introduce Cartesian coordinates. The x- and y-axes are directed along the inclined surface and normal to it; the coordinate $y = 0$ corresponds to the solid surface. We describe the granular material as a continuous medium by the laws of conservation of mass, momentum, and energy of random motion of the granules. In the steady state, all the functions of interest are dependent solely on the y coordinate, and the macroscopic velocity of the flow of particles possesses only one component directed down the inclined surface: $\boldsymbol{u} = \{u(y),\ 0,\ 0\}$. Therefore, the continuity equation is identically satisfied, and the equations for the two components of the momentum and for the root-mean-square velocity v of random motion of the granules, as in [9], are

$$\frac{dp}{dy} = -\rho g \cos\psi, \quad \frac{d}{dy}\Big(\eta \frac{du}{dy}\Big) = -\rho g \sin\psi, \quad \frac{d}{dy}\Big[æ \frac{d}{dy}\Big(\frac{\rho v^2}{2}\Big)\Big] + \eta\Big(\frac{du}{dy}\Big)^2 = I. \tag{1}$$

Here ρ is the density of the granular material, p is the pressure, η is the viscosity, $æ$ is the diffusivity of the energy of random motion, and I is the dissipation rate of random motion of the granules due to their inelastic collisions. The density ρ of the material can be expressed via the density ρ_p of the substance of the granules by the formula $\rho = \rho_p a^3/(a+s)^3$, where a is the granule diameter and s is the mean free path of the granules, so that $a+s$ is the mean distance between the centers of the particles. To obtain closing relations, following [3, 8, 9], we apply the dimensional theory to the desired quantities. For example, the dimensional representation of pressure is the product of the dimensions of mass and acceleration divided by the dimension of area. After each collision, the momentum of a granule changes, in order of magnitude, by mv. Dividing this change in the momentum by the mean time between the collisions t_e, adopting the quantity $(a+s)^2$ as a characteristic area, and representing the particle mass as $m \approx \rho(a+s)^3$, we obtain $p \approx \rho(a+s)v/t_e$. Analogous estimations give

$$p = a_p \rho(a+s)v/t_e, \quad \eta = a_\eta \rho(a+s)^2/t_e, \quad æ = a_æ \rho(a+s)^2/t_e, \quad I = a_I \rho(1-e^2)v^2/t_e, \tag{2}$$

where a_p, a_η, $a_æ$, and a_I are dimensionless multipliers of the order of unity and e is the coefficient of velocity restitution in inelastic collisions.

Following [8, 9], we represent the time elapsing between the collisions as $t_e = t_f + t_c$, where $t_f = s/v$ is the mean free time, $t_c = \alpha a/c$ is the time of intergranular contact, $c = (E/\rho)^{1/2}$ is the velocity of the elastic wave excited in the colliding granules, E is the modulus of elasticity, and α is a dimensionless parameter. If $\alpha = 2$, then the contact time is the time required for the excited elastic wave to execute forward and reverse traveling over the granule diameter. In the case of perfectly rigid granules, their contact time during collisions is infinitely short.

Inserting into (2) the mean time elapsing between collisions and representing the mean free path via the density of the material, we obtain the desired closing relations in the form

$$p = \frac{a_p \rho v^2}{f}, \quad \eta = \frac{a_\eta a \rho_p (\rho/\rho_p)^{2/3} v}{f}, \quad æ = \frac{a_æ a (\rho/\rho_p)^{2/3} v}{f}, \quad I = \frac{a_I(1-e^2)(\rho/\rho_p)^{1/3}\rho v^3}{af}, \tag{3}$$

where $f = 1 - (\rho/\rho_p)^{1/3}(1 - \alpha v/c)$. Thus, the pressure and the dissipative coefficients are expressed as functions of the density of the granules and the velocity of their random motion.

From the first two equations of system (1), it follows that $du/dy = (p/\eta)\tan\psi$. Substituting this expression into the third equation of system (1) and using relations (3), we obtain

$$a_p \frac{d}{dy}\Big(\frac{\rho v^2}{f}\Big) = -\rho g \cos\psi, \quad \frac{d}{dy}\Big[\Big(\frac{\rho}{\rho_p}\Big)^{2/3} \frac{v}{f} \frac{d}{dy}(\rho v^2)\Big] + \frac{2h}{a_æ a^2}\Big(\frac{\rho}{\rho_p}\Big)^{1/3} \frac{\rho v^3}{f} = 0; \tag{4}$$

$$\frac{du}{dy} = \frac{a_p}{a_\eta a}\Big(\frac{\rho}{\rho_p}\Big)^{1/3} v \tan\psi. \tag{5}$$

Here $h = (a_p^2/a_\eta)\tan^2\psi - a_I(1-e^2)$. System (4) allows one to determine the concentration of the granules and the average velocity of their random motion, and Eq. (5) the velocity of the macroscopic motion of the medium.

To formulate the problem, it is required to specify four boundary conditions at the inclined surface and at the free surface of the free-flowing bed as we have two first-order equations for the density and longitudinal velocity and one second-order equation for the thermal velocity. Assuming the inclined surface to have a microrelief, we adopt here the no-slip condition $u(0) = 0$ and the condition $v(0) = v_w$. At the free surface, the flux of thermal energy should vanish, which is equivalent to the condition $(dv/dy)_{y=H} = 0$. As to the boundary condition for the granule concentration, the situation here is less definite. For moderate densities of the granular material (which is the case of interest), the bed is bounded from above by a comparatively narrow transition region between the colliding granules and the granules that move without collisions in the gravitational field. In this region, the particle concentration changes abruptly. The freely travelling granules come back to the region with a higher concentration and bring there momentum and energy, while the particles that escape from the collision region carry momentum and energy over to the region with a lower concentration. In a number of works (see, e.g., [1, 10, 11]), the density at the free surface was assumed to equal zero, but this condition cannot be applied to a medium in which the granules play the main and only part, since the influence of air is ignored and the assumption about the continuity of the medium, which is normally used in various models of granular media, is violated. Therefore, a certain finite density of the material $\rho(H) = \rho_s$ is assumed at the free surface of the bed.

We represent Eqs. (4) and (5) in the nondimensional form by introducing the reference length a and reference velocity v_0:

$$\frac{d}{dy}\Big(\frac{\rho v^2}{f}\Big) = -\frac{\mathrm{R}}{a_p}\,\rho\cos\psi, \qquad \frac{d}{dy}\Big[\frac{\rho^{2/3}v}{f}\,\frac{d}{dy}\,(\rho v^2)\Big] + \frac{2h\rho^{4/3}v^3}{a_{æ}f} = 0; \tag{6}$$

$$\frac{du}{dy} = \frac{a_p}{a_\eta}\,\rho^{1/3}v\tan\psi. \tag{7}$$

Here $f = 1 - \rho^{1/3}(1 - \beta v)$, $\beta = \alpha v_0/c$, and $\mathrm{R} = ag/v_0^2$ is the Richardson number.

To solve system (6) numerically, we represent it in the form of three first-order equations. Performing differentiation in the first equation, we establish the relation between $d\rho/dy$ and dv/dy; then, differentiating the thermal-energy density with respect to the coordinate, we substitute the resultant expression for $d\rho/dy$ into the energy equation. Thus, we obtain the following system of equations:

$$\frac{d\rho}{dy} = -\Big(B + \frac{CD}{2\rho - Cv/A}\Big)\Big/A, \qquad \frac{dv}{dy} = \frac{D}{2\rho - Cv/A}, \qquad \frac{dq}{dy} = -\frac{2h\rho^{4/3}v^3}{a_{æ}f}, \tag{8}$$

where

$$A = v\Big[1 - \frac{2}{3}\,\rho^{1/3}(1 - \beta v)\Big], \quad B = \frac{\mathrm{R}\,f^2\rho\,\cos\psi}{a_p v}, \quad C = \rho(2f - \beta\rho^{1/3}v), \quad D = \frac{fq}{\rho^{2/3}v} + \frac{Bv}{A}.$$

One of the boundary conditions is specified at the inclined surface: $v(0) = v_w$, and two others at the free surface: $\rho(H) = \rho_s$ and $dv(H)/dy = 0$. Using the last two conditions, we find the value of the desired auxiliary quantity q at the free surface:

$$q_s = -\frac{\mathrm{R}\,\rho_s^{5/3}v_s[1 - \rho_s^{1/3}(1 - \beta v_s)]\cos\psi}{a_p[1 - (2/3)\rho_s^{1/3}(1 - \beta v_s)]}, \qquad v_s = v(H).$$

To solve system (8) numerically, we use the shooting method: at the free surface $y = H$, in addition to ρ_s, we specify an arbitrary value of v_s, draw the solution to the plane $y = 0$, compare the obtained value $v(0)$ with the given v_w; if these values do not coincide, we choose a new value of v_s, etc., until the boundary condition for the thermal velocity at the inclined surface becomes fulfilled. Then, from the obtained values of the density of the granules and velocity of their random motion, using Eq. (7), we determine the velocity of the medium down the inclined surface. All first-order equations are solved by the Runge–Kutta method.

We consider the calculation results for the flow of interest for $\psi = 20°$, $e = 0.9$, $v_w = 1$, and $H = 20$ for several values of the Richardson number and the coefficient β that characterizes the contact time of the

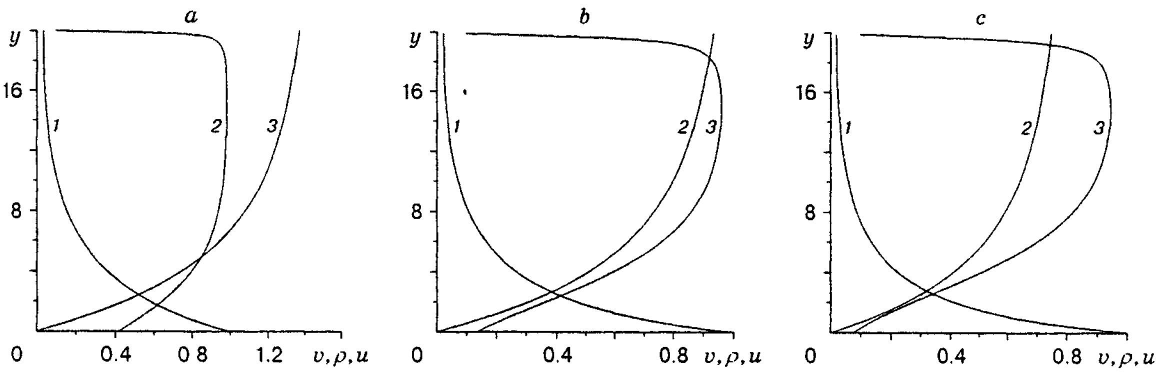

Fig. 1

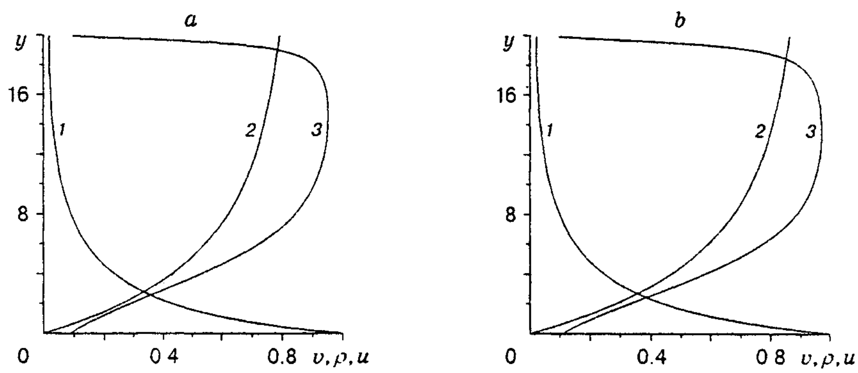

Fig. 2

colliding granules. Figures 1 and 2 show the in-depth (across the bed) distributions of the velocity v of random motion of the granules (curve 1), the density ρ (curve 2), and the macroscopic velocity u of the particles (curve 3). Figure 1a–c corresponds to the case $\beta = 0$ and R = 0.1, 0.02, and 0.01, and Fig. 2a and b to the case $\beta = 0.2$ and 0.5 for R = 0.01. When the Richardson numbers are sufficiently low, a fluidized bed arises on the inclined surface, in which the granules have an appreciable energy of random motion. A decrease in the Richardson number, which, at a constant granule size, corresponds to a rise in the inflow of the thermal energy from the inclined surface into the granular bed, results in an increased thickness of the fluidized bed (Fig. 1). For R = 0.1, 0.02, and 0.01, the ratio of the thickness of the fluidized bed to the total thickness of the granular bed equals $\Delta \approx 0.2H$, $0.3H$, and $0.4H$. At the same time, the density of the granules on the inclined surface decreases: $\rho(0) = 0.42$, 0.13, and 0.07 for R = 0.1, 0.02, and 0.01, which corresponds to the mean free path $s = 0.33$, 0.96, and 1.38, respectively. The density of the substance in the granular bed increases from $\rho(0)$ up to a certain maximum value $\rho_{\max}$ inside the layer, and then decreases to ρ_s. The value $\rho_{\max}$ depends on the Richardson number, decreasing with decreasing R. In all the cases considered, the in-depth distribution of the velocity of random motion of the granules is qualitatively similar: a monotonic growth from v_w at $y = 0$ to a certain value $v_s \approx 0.02$. The macroscopic velocity of the granular flow decreases with decreasing Richardson number: $u_{\max} \approx 1.37$, 0.93, and 0.75 for R = 0.1, 0.02, and 0.01, respectively.

To estimate the effect of the contact time on v, ρ, and u, we carried out the same calculations for R = 0.01 and $\beta = 0.2$ and 0.5 (Fig. 2). An increase in the contact time results in the following changes in the density profiles of the particles in the bed: the density $\rho(0)$ increases with increasing β, the thickness of the fluidized bed decreases, and the macroscopic velocity of the flow rises.

Thus, the study performed has revealed the formation of a fluidized bed in the flow of a granular material down an inclined surface with a relief. The thickness of the fluidized bed increases with intensified inflow of energy to the random motion of the granules. Allowance for the finite contact time, all other conditions being equal, results in a decreased thickness of the fluidized bed and an increased concentration of particles in the immediate vicinity of the inclined surface.

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